Compactness in the Real Numbers, Heine-Borel, Bolzano Weierstrass

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Compactness is a property ascribed to sets that behave nicely in certain ways. We will focus on compactness of sets in Euclidean space (see the compactness in metric spaces document for a more general case). Here, the sets are as the name suggests– small in some sense. This allows us to restrict functions on compact sets and pull many, many interesting properties out of other situations in math when we consider functions on compact sets.

We start with a general, topological definition of compactness. We require the concept of an open cover. Given a set K, $\{U_\alpha\}_{\alpha \in A}$ is an open cover of K if each U_α is open and $\bigcup_A U_\alpha \supset K$.

Definition [0.1] A set $K \subset \mathbb{R}^n$ is called compact if for any open cover $\{U_\alpha\}$ of K, there exists a finite subcollection $U_1, ..., U_m$ which cover K. We usually refer to this as a finite subcover.

Using this definition directly to come up with compact sets is tricky and takes some effort. Finite sets certainly satisfy the finite subcover property, but we will need to investigate more to discover other sets that do. However, coming up with sets which fail the definition is easier.

Non-Examples:

The whole real line $\mathbb{R}$ fails to be compact: we may take an open cover $\{(n, n+2)\}_{n\in\mathbb{Z}}$. The open intervals (a, b) also fail to be compact, by the open cover $\{(a, b - \frac{1}{n}(b-a))\}_{n\in\mathbb{N}}$.

We call a set $M \subset \mathbb{R}^n$ bounded if there exists some large number N so $M \subset B(0, N)$. From the first non-example, we may extrapolate that unbounded sets always fail to be compact, and from the second we may guess that open sets do as well. This is indeed the case.

[0.2] A compact set $K \subset \mathbb{R}^n$ is closed and bounded.

Proof:

Let K be a compact set. Let x be a limit point of K, so there exists $\{x_n\} \to x$ where $x_n \in K$ for $x_n \neq x$. If $x \notin K$, then the sets $U_n = \mathbb{R}^n \backslash B(x, \frac{1}{n})$ form a nested open cover of K. Any finite subcover $U_{i_1}, ..., U_{i_n}$ has $\bigcup U_{i_j} \subset U_N$ for some N. However, since $x_n \to x$, we must have that for some large m, $x_m \in B(x, \frac{1}{N})$ which means $x_m \notin U_N$, contradicting that this covers K. Hence, all limit points of K are contained in K.

Second, $\{B(0, n)\}_{n\in N}$ forms an open cover of $\mathbb{R}^n$ and so an open cover of K. Since some finite subcover covers K, we must have $K \subset B(0, N)$ for some large N. This is the definition of a set being bounded.

Q.E.D.

The interesting fact is that the converse is actually true as well. This is a very famous result.

[0.3] Heine-Borel Theorem: Closed and bounded subsets of $\mathbb{R}^n$ are compact.

Proof:

For readability, we will present the proof on $\mathbb{R}$. The general proof is very similar, only changing some intervals to cubes.

Let K be a set which is both closed and bounded. Let M be such that $K \subset [-M, M]$. By contradiction, assume there is some open cover $\{U_\alpha\}$ which admits no finite subcover of K. Then, at least one of $K \cap [-M, 0]$ or $K \cap [0, M]$ admits no finite subcover either. Let $I_1 = K \cap [a_1, b_1]$ be a piece which does not. Then, again we have that at least one of $K \cap [a_1, \frac{a_1+b_1}{2}]$ or $K \cap [\frac{a_1+b_1}{2}, b_1]$ admits no finite subcover, and we may pick I_2 to be the piece which does not. We repeat this to obtain a sequence of nested closed sets $I_1 \supset I_2 \supset I_3 \supset ...$ such that the diameter of I_n is at most $M2^{1-n}$. Let $x_n \in I_n$, and the sequence x_n is then Cauchy, and so it must converge to some x, where $x \in K$ since K is closed. Using the open cover, we must have $x \in U_\alpha$ for some α, and by the definition of open sets in a metric space, $B(x, \epsilon) \subset U_\alpha$ for some $\epsilon > 0$. However, we then have that I_N is contained in this ball for very large N. Indeed, pick N such that $M2^{1-N} < \epsilon/2$ and $|x - x_N| < \epsilon/2$ to see $I_N \subset B(x, \epsilon)$. This contradicts that I_N has no finite subcover.
Q.E.D.

Despite its namesake, the theorem was first proved by Dirichlet using the implicit assumption that closed, bounded intervals are compact. For a very in-depth history of compactness, see [1].

We now have an easy way to classify things as compact, but what do we gain from it? One strength of compact sets is an easy way to look at singularities, or rather the absence of, for functions.

[0.4] Let K be a compact set and $f : K \to \mathbb{R}^n$ a continuous function. Then, $f(K)$ is compact.

Proof:
Recall that a function is continuous if and only if the preimage of open sets is open. Let $\{U_\alpha\}$ be an open cover of $f(K)$. Then, $\{f^{-1}(U_\alpha)\}$ is an open cover of K. By compactness, there is a finite subcover $\{f^{-1}(U_i)\}_{i=1}^n$, and so $\{U_i\}_{i=1}^n$ covers $f(K)$.
Q.E.D.

In particular, the continuous image of a compact set is bounded. We can use this result to say things like, for any continuous $f : S^1 \to \mathbb{R}$, there is some $M > 0$ so $|f(x)| \leq M$, a useful fact in proofs about functions.

However, there is another motivation for looking at compact sets due to sequences. Historically speaking, this was the original motivation for compactness. The above result, due to Weierstrass, was part of a push by several mathematicians including Dedekind and Cantor to make certain concepts taken for granted previously more rigorous. Bolzano, in a similar vein of thinking about properties like the intermediate value theorem, came up with the greatest lower bound property (which you have likely seen through suprema and infima). Weierstrass used this to show that a bounded infinite set of real numbers has a limit point, which was generalized later. This idea was developed into an early concept of compactness, which we will call sequential compactness (the previous idea we used above, sometimes referred to as *open cover* compactness, was developed later for a more general topological setting) [1].

Definition [0.5] A set K is said to be sequentially compact if any sequence $\{x_n\} \subset K$ has a subsequence $\{x_{n_m}\}$ covering to a point of K.

Being able to start with an arbitrary sequence and get some convergent subsequence is an extraordinarily strong property, and highly useful in analysis proofs that often rely on finding a limit of some sequence. Sequential compactness and open cover compactness are not equivalent properties in general topological spaces. However, in $\mathbb{R}^n$ (and all metric spaces for that matter), they are. The proof of this is a distilled version of the combined work of Bolzano and Weierstrass.

[0.6] Bolzano-Weierstrass Theorem: A subset of $\mathbb{R}^n$ is sequentially compact if and only if it is closed and bounded.

Proof:
We prove this statement by showing that failing to be sequentially compact is equivalent to being nonclosed or being unbounded. First, we prove the reverse direction.

If K is an unbounded set, then given any M we may find $x \in K$ so $d(0, x) > M$. Thus, we may recursively define a sequence $\{x_n\} \subset K$ so $d(0, x_{n+1}) \geq 1 + d(0, x_n)$ for all n. By the triangle inequality, $d(x_n, x_m) \geq 1$ for $n \neq m$, and no subsequence converges.

If K is a nonclosed, bounded set, then K fails to contain some limit point and there is some $x \notin K$ so $\{x_n\} \subset K$ converges to x. Any subsequence of $\{x_n\}$ converges to x, so no subsequence can converge to a point in K.

Second, let K fail to be sequentially compact. Then, there exists $\{x_n\} \subset K$ such that no subsequence converges to a point of K. We consider two cases: one where a subsequence does converge and the other where no subsequence converges.

First, assume that some subsequence converges to some $y \in \mathbb{R}^n$. Then, y is a limit point of K, but $y \notin K$ implies K is not closed.

Next, assume that no subsequence converges. Then, for any $x \in \mathbb{R}^n$ there exists some $\epsilon_x > 0$ and some N_x so $x_n \notin B(x, \epsilon_x)$ for all $n > N_x$. Since $\overline{B(0, M)}$ is compact (by the Heine-Borel theorem), and these $B(x, \epsilon_x)$ cover $\overline{B(0, M)}$, there is a finite subcover with centers $x_1, ..., x_{i_M}$. Let $N_M = \max\{N_1, ..., N_{i_M}\}$, and we have that for $n > N_M$, $d(0, x_n) > M$. Since we may do this for all M, K must be unbounded.

Q.E.D.

Exercise 1: Prove the Bolzano-Weierstrass Theorem on $\mathbb{R}$ without using the Heine-Borel theorem or open cover compactness (the part of the proof that differs from the above is showing that closed and bounded implies sequentially compact. Show that a bounded sequence in $\mathbb{R}$ has a limit point by using the $\liminf$ or $\limsup$. This is why Bolzano's observation about the existence of an infimum lead to this theorem).

Exercise 2: Let $\{K_n\}$ be a collection of compact sets so $K_n \supset K_{n+1}$. Show that $\cap_{i=1}^{\infty} K_i$ is nonempty (this is known as the nested compact set property, and you can also furnish a proof of the previous exercise or above theorem using this property).

References

[1] Manya Raman-Sundstrom. A pedagogical history of compactness, 2014.